

Approach to the solution of the complete intersection problem for set partitions

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Abstract

We prove the Complete t - intersection problem for set partitions of $[n]$ when $t \leq n/3$. To extend the proof for the larger t it is necessary to consider analytical property of the explicit function.

I Introduction

Let $\mathcal{P}(n)$ be the set of partitions of set $[n]$. We say that two partitions $p_1, p_2 \in \mathcal{P}(n)$ are t - intersect if they have at least t common parts. The main problem we consider here is obtaining the maximal cardinality of t - intersection family from $\mathcal{P}(n)$. We say that i is fixed in the partition $p \in \mathcal{P}(n)$ if it contains $\{i\}$. Denote $f(p)$, $p \in \mathcal{P}(n)$ the set of points from $[n]$ fixed by p .

Denote $\Omega_t(n)$ the family of t - intersection sets from $\mathcal{P}(n)$. Let

$$M(n, t) = \max\{|\mathcal{A}| : \mathcal{A} \subset \Omega_t(n)\}.$$

Main result of this work is the proof of the following

Theorem 1 *Let $n/3 \geq t \geq 2$, then*

$$M(t, n) = \max_{r \in [0, \lfloor (n-t)/2 \rfloor]} |\{p \in \mathcal{P}(n) : [t + 2r] \cap f(p) \geq t + r\}|.$$

Our proof extend the ideas from the work [3], where was solved the complete intersection theorem for set of permutations and t - intersection in that work considered as coincidence by common cycles. In [1] was proved the particular case of intersection problem for set partitions. There was proved, that for sufficiently large $n > n_0(t)$

$$M(t, n) = B(n - t),$$

where $B(n)$ is Bell number, the number of elements in $\mathcal{P}(n)$.

Also from [1] we know that ($n \geq 2$)

$$M(1, t) = B(n - 1).$$

Thus our Theorem deliver the complete solution of t - intersection problem for the set partitions when $t \leq n/3$.

Let's, make some definitions and consider some relations which we will use later.

Bell numbers $B(n)$ satisfy the following relations

$$B(n) = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!}, \quad (1)$$

$$B(n) = \sum_{i=0}^n \binom{n}{i} B(i).$$

Also we need numbers $\tilde{B}(n)$, which are number of elements in $\mathcal{P}(n)$ which do not have singletons. They satisfies relations

$$\tilde{B}(n+1) = \sum_{i=0}^{n-1} \binom{n}{i} \tilde{B}(i), \quad (2)$$

$$\tilde{B}(n) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} B(i) \quad (3)$$

Now we return to the proof of main problem.

II Proof of the main Theorem

Define fixing procedure $F(i, j, p), i \neq j \in [n]$ on the set of partitions $p \in \mathcal{P}(n)$ ([1]) :

$$F(i, j, p) = \begin{cases} p, & i \in p' \in p, j \notin p', \\ \tilde{p} \cup \{i\}, & \{i, j\} \subset p' \in p \end{cases}$$

where \tilde{p} is partition, obtained from p by excluding point i from corresponding part of p .

Then fixing operator on the set $\mathcal{A} \subset \Omega_t(n)$ is defined as follows ($p \in \mathcal{A}$)

$$F(i, j, p, \mathcal{A}) = \begin{cases} F(i, j, p), & F(i, j, p) \notin \mathcal{A}, \\ p, & F(i, j, p) \in \mathcal{A}. \end{cases}$$

At last define operator

$$\mathcal{F}(i, j, \mathcal{A}) = \{F(i, j, p, \mathcal{A}); p \in \mathcal{A}\}$$

It is easy to see, that fixing operator $\mathcal{F}(i, \mathcal{A})$ preserve volume of \mathcal{A} and its t -intersection property. At last note, that making shifting operation finitely number of times we obtain from the set \mathcal{A} compressed set, which has the property that for all $i \neq j, \in [n]$

$$\mathcal{F}(i, j, \mathcal{A}) = \mathcal{A}$$

and arbitrary pair of partitions p_1, p_2 from the compressed \mathcal{A} intersect by at least t fixed points.

Next define (usual) shifting procedure $L(v, w, p)$, $1 \leq v < w \leq n$ as follows. Let $p = \{\{j_1, \dots, j_{q-1}, v, j_{q+1}, \dots, j_s\}, \dots, \{w\}, \pi_1, \dots, \pi_c\} \in \mathcal{A}$, then

$$L(v, w, p) = \{j_1, \dots, j_{q-1}, w, j_{q+1}, \dots, j_s\}, \dots, \{v\}, \pi_1, \dots, p_s\}.$$

If $p \in \mathcal{A}$ does not fix v we set

$$L(v, w, p) = p.$$

Now define shifting operator $L(v, w, p, \mathcal{A})$ as follows

$$L(v, w, p, \mathcal{A}) = \begin{cases} L(v, w, p), & L(v, w, p) \notin \mathcal{A}, \\ p, & L(v, w, p) \in \mathcal{A}. \end{cases}$$

At last we define operator $\mathcal{L}(v, w, \mathcal{A})$:

$$\mathcal{L}(v, w, \mathcal{A}) = \{L(v, w, p, \mathcal{A}); p \in \mathcal{A}\}.$$

It is easy to see that operator $\mathcal{L}(v, w, \mathcal{A})$ does not change the volume of \mathcal{A} and preserve t - intersection property. Also it is easy to see that after finite

number of operations we come to the compressed t - intersection set \mathcal{A} of the volume for which

$$L(v, w, \mathcal{A}) = \mathcal{A}, \quad 1 \leq v < w \leq n$$

and each pair of permutations form \mathcal{A} t - intersect by fixed elements. Next we consider only such sets \mathcal{A} .

Let $\mathcal{D}(v, w, \mathcal{A})$ be the same operator as $\mathcal{L}(v, w, \mathcal{A})$ but without condition $v < w$ (but still $v \neq w$).

Denote

$$\beta(\ell) = \frac{\sum_{j=0}^{n-\ell+1} \binom{n-\ell+1}{j} \tilde{B}\left(n - \frac{\ell+t}{2} + 1 - j\right)}{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} - j\right)}.$$

need the following

Lemma 1 *Let $|\mathcal{A}| = M(n, t)$,*

$$D(v, w, \mathcal{A}) = \mathcal{A}, \quad \forall v \neq w \leq \ell$$

and for $\ell = t + 2r$ the inequality

$$\frac{\ell + 1}{\frac{\ell-t}{2} + 1} > \beta(\ell + 2) \tag{4}$$

holds. Then $\mathcal{D}(v, w, \mathcal{A}) = \mathcal{A} \quad \forall v \neq w \leq \ell + 2$.

Let

$$|\mathcal{A}| = M(t, n)$$

and \mathcal{A} is invariant under shifting and fixing operators . Let

$$\mathcal{D}(v, w, \mathcal{A}) = \mathcal{A}, \quad 1 \leq v, w \leq \ell,$$

but

$$\mathcal{A} \neq \mathcal{D}(v, \ell + 1, \mathcal{A})$$

for some $v \in [\ell]$.

We set

$$\mathcal{A}' = \{p \in \mathcal{A} : D(v, \ell + 1, p) \notin \mathcal{A}, v \in [\ell]\}$$

We identify $2^{[n]}$ with the family of subsets of $[n]$.

Define

$$B(\mathcal{A}) = \{f(p); p \in \mathcal{A}\} \subset 2^{[n]}.$$

It is easy to see, that the set $B(\mathcal{A})$ is upper ideal with the order induced by inclusion. Denote $M(\mathcal{A})$ the set of minimal elements of $B(\mathcal{A})$.

Let

$$s^+(M(\mathcal{A})) = \max_{M \in M(\mathcal{A})} s^+(M),$$

where

$$s^+(M) = \max_{i \in M} i.$$

We also need one more lemma.

Lemma 2 *Let $|\mathcal{A}| = M(n, t)$, $s^+(M(\mathcal{A})) = \ell$. Then*

$$\frac{\ell - t}{2(\ell - 1)} \beta(\ell) \leq 1. \quad (5)$$

Later we show that there exists unique ℓ satisfies inequalities (4) and (5). From this follows statement of the Theorem.

First we prove 1. Assume that $\mathcal{A}' \neq \emptyset$.

Let

$$\mathcal{A}(i) = \{p \in \mathcal{A}' : |f(p) \cap [\ell]| = i\}.$$

It follows that $\mathcal{A}(i) \neq \emptyset$ for some $i \in [\ell]$.

Let

$$\mathcal{A}'(i) = \{f(p) \cap [\ell + 2, n]; p \in \mathcal{A}(i)\}.$$

Denote

$$\mathcal{B}(i) = \{p \in \mathcal{P}(n) : |B(p) \cap [\ell]| = i - 1, \ell + 1 \in p, f(p) \cap [\ell + 2, n] = \mathcal{A}'(i)\}.$$

We have

$$|\mathcal{A}(i)| = \binom{\ell}{i} \sum_{p \in \mathcal{A}'(i)} \tilde{B}(n - i - |f(p)|),$$

$$|\mathcal{B}(i)| = \binom{\ell - 1}{i - 1} \sum_{p \in \mathcal{A}'(i)} \tilde{B}(n - i - |f(p)|)$$

and for any $i \in [\ell + 1]$, we have

$$\mathcal{C}(i) = (\mathcal{A} \setminus \mathcal{A}(i)) \cup \mathcal{B}(\ell + t - i) \in \Omega_t(n).$$

Next we demonstrate, that if $\mathcal{A}(i) \neq \emptyset$ and (at first) $i \neq \frac{\ell+t}{2}$, then

$$\max\{|\mathcal{C}(i)|, |\mathcal{C}(\ell + t - i)|\} > |\mathcal{A}| \quad (6)$$

and come to contradiction of the maximality of \mathcal{A} .

Suppose that (6) is not valid, then

$$\begin{aligned} \binom{\ell}{i-1} \sum_{p \in \mathcal{A}'(i)} \tilde{B}(n-i-|f(p)|) &\leq \binom{\ell}{\ell+t-i} \sum_{p \in \mathcal{A}'(\ell+t-i)} \tilde{B}(n-(\ell+t-i)-|f(p)|), \\ \binom{\ell}{\ell+t-i-1} \sum_{p \in \mathcal{A}'(\ell+t-i)} \tilde{B}(n-(\ell+t-i)-|f(p)|) &\leq \binom{\ell}{i} \sum_{p \in \mathcal{A}'(i)} \tilde{B}(n-i-|f(p)|). \end{aligned}$$

We have $\mathcal{A}(i) \neq \emptyset$ hence $\mathcal{A}(\ell+t-i) \neq \emptyset$ and

$$i(\ell+t-i) \leq (\ell-i+1)(i+1-t).$$

Because $t \geq 2$ the last inequality is false. This contradiction shows that $\mathcal{A}(i) = \emptyset$ for $i \neq \frac{\ell+t}{2}$.

Consider now that $2 | (\ell+t)$. Next we demonstrate that if (4) is true, then $\mathcal{A}\left(\frac{\ell+t}{2}\right) = \emptyset$.

We have

$$\left| \mathcal{A}\left(\frac{\ell+t}{2}\right) \right| = \binom{\ell}{\frac{\ell+t}{2}} \sum_{p \in \mathcal{A}\left(\frac{\ell+t}{2}\right)} \tilde{B}(n-i-|f(p)|).$$

Next introduce set

$$\begin{aligned} \mathcal{C} = \left\{ p \in \mathcal{P}(n) : |f(p) \cap [\ell]| = \frac{\ell+t}{2} - 1; \ell+1, n \in f(p), \right. \\ \left. f(p) \cap [\ell+2, n] \in \mathcal{A}'\left(\frac{\ell+t}{2}\right) \right\}. \end{aligned}$$

Denote

$$\mathcal{D} \left\{ p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right) : \{n\} \in p \right\}.$$

It is easy to see, that

$$\mathcal{G} = \left\{ \mathcal{A} \setminus \left\{ p \in \mathcal{A}\left(\frac{\ell+t}{2}\right) : \{n\} \notin f(p) \right\} \cup \mathcal{C} \right\} \subset \Omega_t(n).$$

Next we demonstrate, that if $\mathcal{A}\left(\frac{\ell+t}{2}\right) \neq \emptyset$, and (4) is true, then we come to the contradiction of the maximality of \mathcal{A} :

$$|\mathcal{G}| > |\mathcal{A}|. \tag{7}$$

We have

$$|\mathcal{C}| = \binom{\ell}{\frac{\ell+t}{2}-1} \sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right), \{n\} \in f(p)} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right).$$

Inequality (7) is equivalent to

$$\begin{aligned} & \binom{\ell}{\frac{\ell+t}{2}-1} \sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right), \{n\} \in p} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right) \\ & > \binom{\ell}{\frac{\ell+t}{2}} \sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right), \{n\} \notin p} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right) \\ & = \binom{\ell}{\frac{\ell+t}{2}} \sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right)} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right) \\ & \quad - \sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right), \{n\} \in p} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right). \end{aligned}$$

From here we have

$$\begin{aligned} & \binom{\ell+1}{\frac{\ell+t}{2}} \sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right), \{n\} \in p} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right) \\ & > \binom{\ell}{\frac{\ell+t}{2}} \sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right)} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right). \end{aligned}$$

Hence

$$\frac{\ell+1}{\frac{\ell-1}{2}+1} > \beta_1(\ell) \triangleq \frac{\sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right)} \tilde{B}\left(n - \frac{\ell+t}{2} - |f(p)|\right)}{\sum_{p \in \mathcal{A}'\left(\frac{\ell+t}{2}\right), \{n\} \in p} \tilde{B}\left(n - \frac{\ell-t}{2} - |f(p)|\right)}.$$

Let's prove that

$$\beta(\ell+2) \geq \beta_1(\ell). \tag{8}$$

If proved, then it follows that (4) is true and we need the condition of Lemma 1 only for $|\mathcal{A}|$ to be maximal. Thus to complete the proof of Lemma 1 we need to prove inequality (8).

Inequality (8) is the consequence of FKG inequality. Indeed, consider more strong inequality

$$\frac{\sum_{X \in \Gamma} \tilde{B}\left(n - \frac{\ell+t}{2} - |X|\right)}{\sum_{X \in \Gamma, x \in X} \tilde{B}\left(n - \frac{\ell+t}{2} - |X|\right)} \leq \frac{\sum_{X \in 2^{[n-(\ell+t)/2-1]}} \tilde{B}\left(n - \frac{\ell+t}{2} - |X|\right)}{\sum_{X \in 2^{[n-(\ell+t)/2-1]}, x \in X} \tilde{B}\left(n - \frac{\ell+t}{2} - |X|\right)} \quad (9)$$

where $\Gamma \subset 2^{[n-(\ell+t)/2-1]}$ is upper ideal and $x \in [n - (\ell + t)/2 - 1]$. Remind, that FKG inequality says that for $\mu : 2^{[m]} \rightarrow R_+$, such that

$$\mu(a)\mu(b) \leq \mu(a \cap b)\mu(a \cup b), \quad a, b \in 2^{[m]}, \quad (10)$$

and for pair of nondecreasing functions $f_1, f_2 : 2^{[m]} \rightarrow R$ the following inequality is valid

$$\sum_{Y \in 2^{[m]}} \mu(Y) f_1(Y) \sum_{Y \in 2^{[m]}} \mu(Y) f_2(Y) \leq \sum_{Y \in 2^{[m]}} \mu(Y) f_1(Y) f_2(Y) \sum_{Y \in 2^{[m]}} \mu(Y). \quad (11)$$

Now we choose

$$\mu(Y) = \tilde{B}\left(n - \frac{\ell+t}{2} - |Y|\right). \quad (12)$$

Note, that if (10) is true for this choice of μ , then setting $f_1 = I_{X \in \Gamma: x \in X}$, $f_2 = I_{X \in 2^{[n-(\ell+t)/2-1]}, x \in X}$ in (11) prove inequality (23). Now we prove, that μ from (12) satisfies (10).

From (1) and (3) follows formula

$$\tilde{B}(n) = \frac{1}{e} \sum_{i=1}^{\infty} \frac{(i-1)^n}{n!}. \quad (13)$$

Then we should prove that for $a, b \geq 0, \delta \leq \min a, b$,

$$\tilde{B}(n-a)\tilde{B}(n-b) \leq \tilde{B}(n-\delta)\tilde{B}(n-a+b-\delta) \quad (14)$$

Using formula (21) it is easy to see that inequality (14) follows from the inequality

$$i^{n-a}j^{n-b} + i^{n-b}j^{n-a} \leq i^{n-\delta}j^{a+b-\delta} + j^{n-\delta}i^{a+b-\delta}. \quad (15)$$

which can be easily verified proving the convexity of the function on δ in the RHS of (15) in the interval $[a, b]$ by differentiation.

Next we prove Lemma 2. Define

$$M_0(\mathcal{A}) = \{E \in M(\mathcal{A}); s^+(E) = s^+(M(\mathcal{A})) = \ell\}$$

and

$$M_1(\mathcal{A}) = M(\mathcal{A}) \setminus M_0(\mathcal{A}).$$

It is easy to see, that for $E_1 \in M_0(\mathcal{A}), E_2 \in M_1(\mathcal{A})$,

$$|(E_1 \setminus \{\ell\}) \cap E_2| \geq t$$

and for $E_1, E_2 \in M_0(\mathcal{A}), |E_1 \cap E_2| = t$,

$$|E_1| + |E_2| = \ell + t.$$

Put

$$M_0(\mathcal{A}) = \cup_i R(i),$$

where

$$R(i) = M_0(\mathcal{A}) \cap \binom{[n]}{i}.$$

Denote

$$R'(i) = \{E \setminus \{\ell\}; E \in R(i)\}.$$

Next we are going to prove, that if (4) is not true, then $R(i) = \emptyset$.

Suppose, that $R(i) \neq \emptyset$ for some i . At first assume, that $i \neq \frac{\ell+t}{2}$.

For $C \subset 2^{[n]}$ with the set of minimal elements M denote $V(C) \subset \mathcal{P}(n)$ the complete set of partitions for which M is a set of minimal elements of an upper ideal of the set of fixed points.

Denote

$$\begin{aligned} F_1 &= M_1(\mathcal{A}) \cup (M_0(\mathcal{A}) \setminus (R(i) \cup R(\ell+t-i))) \cup R'(i), \\ F_2 &= M_1(\mathcal{A}) \cup (M_0(\mathcal{A}) \setminus (R(i) \cup R(\ell+t-i))) \cup R'(\ell+t-i). \end{aligned}$$

It is easy to see that for $E_1, E_2 \in F_i, |E_1 \cap E_2| \geq t$ and thus $F(F_1), V(F_2) \in \Omega_t(n)$. We are going to show, that if $R(i) \neq \emptyset$, then

$$\max\{|V(F_1)|, |V(F_2)|\} > |\mathcal{A}| \quad (16)$$

which gives us contradiction.

We have

$$|\mathcal{A}| \setminus V(F_1) = |R(\ell+t-i)| = \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-\ell-t+i-j) \quad (17)$$

and

$$|V(F_1) \setminus \mathcal{A}| = |R(i)| \sum_{j=0}^{n-t} \binom{n-\ell}{j} \tilde{B}(n-i-j+1). \quad (18)$$

Also

$$|\mathcal{A} \setminus V(F_2)| = |R(i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-i-j), \quad (19)$$

$$|V(F_2) \setminus \mathcal{A}| = |R(\ell+t-i)| \sum_{j=0}^{n-\ell} \tilde{B}(n-\ell-t+i-j+1). \quad (20)$$

If (16) is not true, then from (17)-(20) it follows

$$\begin{aligned} |R(i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-i-j+1) &\leq |R(\ell+t-i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-\ell-t+i-j), \\ |R(\ell+t-i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-\ell-t+i-j+1) &\leq |R(i)| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}(n-i-j). \end{aligned}$$

These inequalities couldn't be valid together due to monotonicity of $\tilde{B}(n)$.

Now consider the case $i = \frac{\ell+t}{2}$. We are going to prove, that if inequality (5) is not true, then $R\left(\frac{\ell+t}{2}\right) = \emptyset$. Simple averaging argument shows that there exists $i \in [\ell-1]$ and $Z \subset R'\left(\frac{\ell+t}{2}\right)$ such that $i \in E$ for all $E \in Z$ and

$$|Z| \geq \frac{\ell-t}{2(\ell-1)} \left| R'\left(\frac{\ell+t}{2}\right) \right|. \quad (21)$$

Because $|E_1 \cap E_2| \geq t$, $E_1, E_2 \in Z$ and $R(i) = \emptyset$, $i \neq \frac{\ell+t}{2}$ then for all $E_1, E_2 \in D$, where

$$D = \left(M(\mathcal{A}) \setminus R\left(\frac{\ell+t}{2}\right) \right) \cup Z$$

we have $|E_1 \cap E_2| \geq t$. Hence $V(D) \in \Omega_t(n)$ and we left to show that if (5) is not true, then

$$|V(D)| > |\mathcal{A}|. \quad (22)$$

Consider the partition

$$\begin{aligned}\mathcal{A} &= V(M(\mathcal{A})) = S_1 \cup S_2, \\ S_1 &= V\left(M(\mathcal{A}) \setminus R\left(\frac{\ell+t}{2}\right)\right), \\ S_2 &= V\left(R\left(\frac{\ell+t}{2}\right)\right) \setminus S_1\end{aligned}$$

and partition

$$\begin{aligned}V(D) &= S_1 \cup S_3, \\ S_3 &= V(D) \setminus S_1.\end{aligned}$$

Then (22) is equivalent to

$$|S_3| > |S_2|.$$

It is easy to show that

$$\begin{aligned}|S_2| &= \left|R\left(\frac{\ell+t}{2}\right)\right| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} - j\right), \\ |S_3| &= |Z| \sum_{j=0}^{n-\ell+1} \binom{n-\ell+1}{j} \tilde{B}\left(n - \frac{\ell+t}{2} + j\right).\end{aligned}\tag{23}$$

Using (21) and (23) we conclude that

$$\begin{aligned}& \frac{\ell-t}{2(\ell-1)} \left|R\left(\frac{\ell+t}{2}\right)\right| \sum_{j=0}^{n-\ell+1} \binom{n-\ell+1}{j} \tilde{B}\left(n - \frac{\ell+t}{2} + 1 - j\right) \\ & > \left|R\left(\frac{\ell+t}{2}\right)\right| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} - j\right),\end{aligned}$$

Last inequality is equivalent to the opposite inequality to (5) (if $R\left(\frac{\ell+t}{2}\right) \neq \emptyset$.) But from here follows the contradiction of the maximality of \mathcal{A} . Thus (5) holds.

Now we rewrite inequality (4) as

$$\ell + 2 < t + 2 \frac{t-1}{\beta(\ell+2)-2},$$

and inequality (5) as

$$\ell \leq t + 2 \frac{t-1}{\beta(\ell)-2}.$$

It is left to show that from inequality function

$$\varphi(\ell) = t - \ell + 2 \frac{t-1}{\beta(\ell)-2}$$

change sign in the interval $[t, n]$ not more than one time. We will not try to prove (or disprove) this, but only show that φ has this property if $t \leq n/3$. To prove this we first show that φ is \cup -convex on interval $[t, n]$. Obviously $\varphi(t) > 0$ and when $t \leq n/3$, then $\varphi(n) < 0$. From these fact will follow the statement of the Theorem 1 .

We have

$$\begin{aligned} \beta(\ell) &= \frac{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} + 1 - j\right) + \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} - j\right)}{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} - j\right)} \\ &= 1 + \frac{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} + 1 - j\right)}{\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} - j\right)}. \end{aligned}$$

Now using identity (21) we derive the relations

$$\begin{aligned} \gamma(n, \ell, t) &= \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} - j\right) \\ &= \sum_{i=1}^{\infty} \frac{(-1)^{n-\frac{\ell+t}{2}} (i-1)^{n-\frac{\ell+t}{2}}}{i!} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} (-1)^j (i-1)^{-j} \\ &= \sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}} (i-2)^{n-\ell}}{i!}. \end{aligned}$$

Similar calculations show the validity of the following identity

$$\begin{aligned} \gamma(n+2, \ell+2, t) &= \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \tilde{B}\left(n - \frac{\ell+t}{2} + 1 - j\right) \\ &= \sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}+1} (i-2)^{n-\ell}}{i!}. \end{aligned}$$

Hence we have for $\beta(\ell) - 2$ the expression

$$\begin{aligned}\beta(\ell) - 2 &= \frac{\gamma(n+2, \ell+2, t)}{\gamma(n, \ell, t)} = \frac{\sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}+1} (i-2)^{n-\ell}}{i!}}{\sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}} (i-2)^{n-\ell}}{i!}} - 1 \\ &= \frac{\sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}} (i-2)^{n-\ell+1}}{i!}}{\sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}} (i-2)^{n-\ell}}{i!}}.\end{aligned}$$

We obtain the following expression for the function $\varphi(\ell)$:

$$\varphi(\ell) = t - \ell + 2(t-1) \frac{\sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}} (i-2)^{n-\ell}}{i!}}{\sum_{i=2}^{\infty} \frac{(i-1)^{\frac{\ell-t}{2}} (i-2)^{n-\ell+1}}{i!}}.$$

It is easy to show that second derivative of this function is negative, this complete the proof of the Theorem 1 .

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